Calabi-Yau Pairs of Complexity Zero

Work in progress w/ F. Figueroa

Outline

1 - Background

Work over $\mathbb{C}$.
All varieties are normal & $\mathbb{Q}$-factorial by default.

Definitions

- A "log Calabi-Yau pair" (logCY) is a pair (X,B) which is log canonical (lc) w/ $K_X+B \sim_{\mathbb{Q}} 0$.
- The "complexity" of a pair (X,B) is

$$c(X,B) = \dim(X) + \rho(X) - |B|.$$

Examples

1) If (X, B) is "toric log CY" then

(X, B) is log CY (w/ $K_X + B \sim 0$) &

$$c(X, B) = 0.$$

2) If X is Calabi-Yau ($K_X \sim_{\mathbb{Q}} 0$) then

$$c(X) = c(X, 0)$$
$$= \dim(X) + \rho(X).$$

3) If X Fano & $-mK_X$ very ample, for $D \in |-mK_X|$ general $(X, \frac{1}{m}D)$ is log CY &

$$c(X, \tfrac{1}{m}D) = \dim(X) + \rho(X) - \tfrac{1}{m}$$

Theorem (Brown-McKernan-Svaldi-Zong '18)

- If (X,B) is lc & $-(K_X+B)$ is nef, then $c(X,B) \geq 0$.
- If, in adittion, $c(X,B) < 1$, then there is a toric log Calabi-You pair (X,Δ) w/ $\lfloor B \rfloor \leq \Delta$.

Remarks

- Conjectured by Shokurov in 2000.
- Related to Kobayashi-Ochiai's characterization of proj. spaces in terms of "Fano index."

Applications and Connections

i) Polarized Endomorphisms
(Meng - Zhang '20, Meng - Zhong '23)

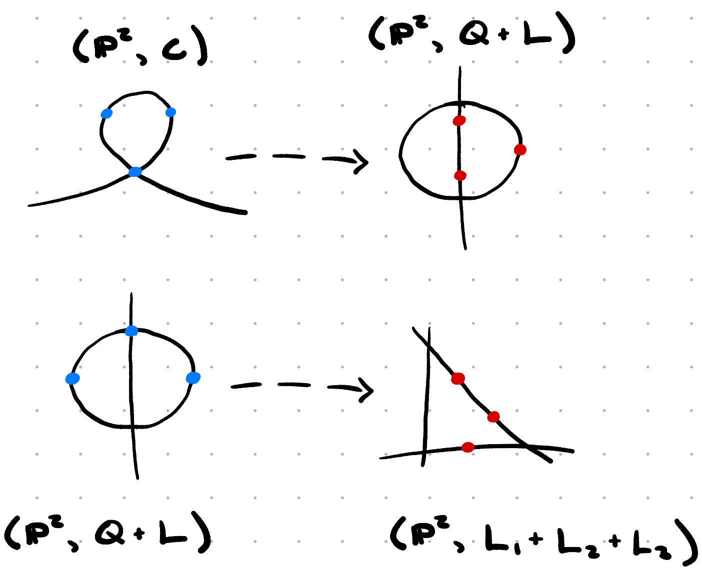
ii) Minimal Log Discrepancies
(Moraga '23)

iii) Mirror Symmetry
(Gross - Hacking - Keel '15)

iv) Dual Complexes
(Mauri '20, Mauri - Moraga '24)

One More Definition

- Let (X,B) log CY pair.
  Its "birational complexity" is
  $$c_{bir}(X,B) = \inf\{c(X',B') \mid (X',B') \simeq_{crep} (X,B)\}.$$



2 - Main Theorems

Theorem (Brown-McKernan-Svaldi-Zong '18)

Let (X,B) be a log CY pair w/ $c(X,B)=0$.
Then there is a toric log CY pair (X,Δ) w/ $\lfloor B \rfloor \leq \Delta \leq \lceil B \rceil$.

Theorem 1 (E.-Figueroa, In progress)

Let (X,B) be log CY w/ $c(X,B)=0$.
There are toric log CY $(X,\Delta_1), \ldots, (X,\Delta_r)$
& nonnegative $b_1, \ldots, b_r \in [0,1] \cap \mathbb{Q}$ w/ $\sum_{i=1}^r b_i = 1$ s.t.

$$B = \sum_{i=1}^r b_i \Delta_i .$$

Theorem (Mauri-Moraga '24)

Let (X,B) be a log CY pair w/ $K_X + B \sim 0$.
Then $c_{bir}(X,B) = 0$ if & only if there is a crepant birational map $(\mathbb{P}^n, H_0 + \ldots + H_n) \dashrightarrow (X,B)$.

Theorem 2 (E.-Figueroa, In progress)

Let (X,B) be log CY.
Then $c_{bir}(X,B) = 0$ if & only if there is a crepant birational map

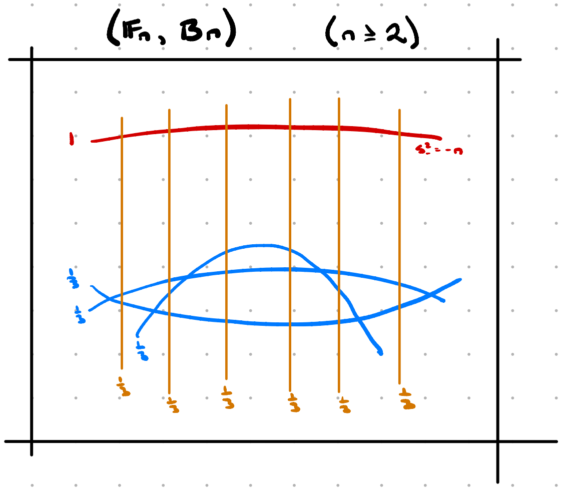
$$\varphi : (Y, B_Y) \dashrightarrow (X,B)$$

s.t.:

1) $c(Y, B_Y) = 0$,
2) Y is a "generalized Bott tower."

3 - Examples

Optimality of Theorem 2



<u>Key Properties</u>

1) $(\mathbb{F}_n, B_n)$ log CY w/ complexity zero.

2) B_n SNS

3) S_n^- is unique lc place; pair is <u>terminal</u> away from S_n^-.

4) (2) & (3) hold after smooth blow-ups.

- Consider crepant $\varphi : (\mathbb{F}_n, B_n) \dashrightarrow (X, B)$ w/ $X \cong \mathbb{P}^2, \mathbb{F}_m$ $(m \geq 0)$.

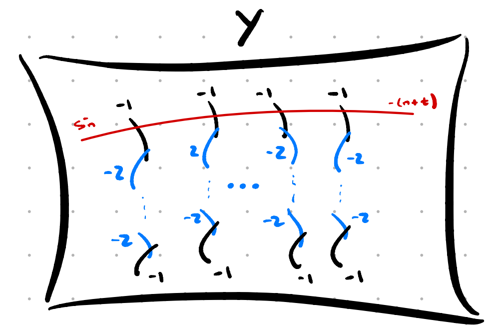
★ <u>Claim:</u> $X \cong \mathbb{F}_{n+r}$ w/ $r \geq 0$, $B \neq B_{n+r}$.

- Blow-up centers of φ^{-1}-exceptional curves to resolve:

$$\begin{array}{ccc} & Y & \\ {}^{f}\swarrow & & \searrow^{g} \\ \mathbb{F}_n & \overset{\varphi}{\dashrightarrow} & X \end{array}$$

- All φ^{-1}-exc curves are non-terminal places of $(\mathbb{F}_n, B_n)$.

$\rightsquigarrow$ f just blows up points on $\bar{s}_n$.



- Say f blows up t points.

Then g blows up t or $t+1$ points.

- $\bar{s}_n$ part of anti-canonical cycle at all steps

$\Rightarrow$ $\bar{s}_n \cdot C = \{0, 1\}$ whenever C is (-1)-curve

$$\Rightarrow (\varphi_* \bar{s}_n)^2 \leq -(n+t) + t + 1$$
$$= -n+1$$
$$\leq -1$$

$\Rightarrow$ $X \ncong \mathbb{P}^2$

$\Rightarrow$ $\rho(X) = 2$

$\Rightarrow$ g blows up t pts

$\Rightarrow$ $(\varphi_* \bar{s}_n)^2 \leq -n$

$\Rightarrow$ $X \cong \mathbb{F}_{n+r}$ w/ $r \geq 0$.

Motivation for Proof of Theorem 1

- Idea:

Obtain toric log CY (X,Δ) w/ $\lfloor B \rfloor \leq \Delta \leq \lceil B \rceil$.

Define

$$\lambda_1 = \lambda_1(X, B; \Delta)$$
$$= \max\{\lambda \in [0,1] \mid \lambda\Delta \leq B\}.$$

Two cases:

1) $\lambda_1 = 1 \implies B = \Delta$ & (X,B) toric log CY.

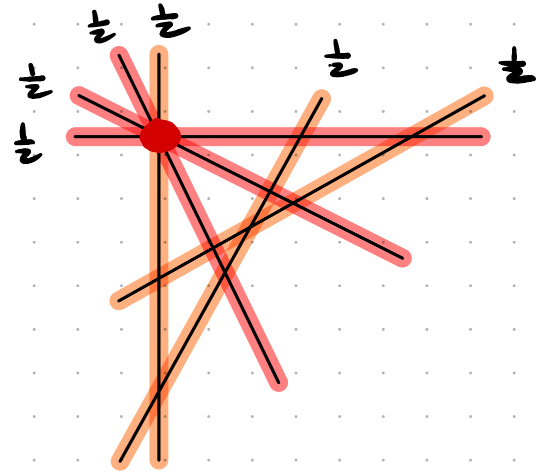
2) $\lambda_1 < 1 \implies B' := \dfrac{B - \lambda_1 \Delta}{1 - \lambda_1}$ nonzero,

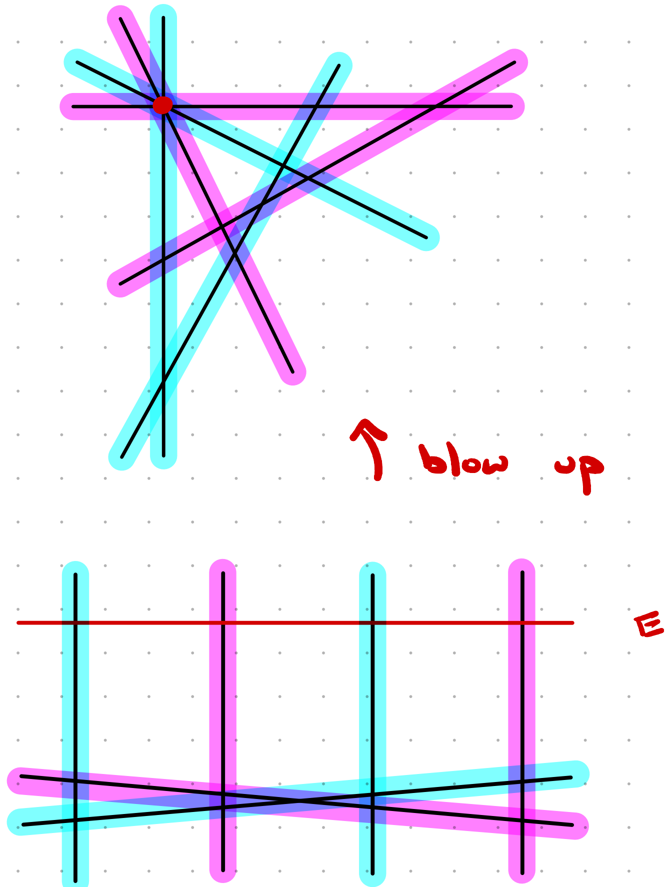
$B' \sim_{\mathbb{Q}} -K_X$,

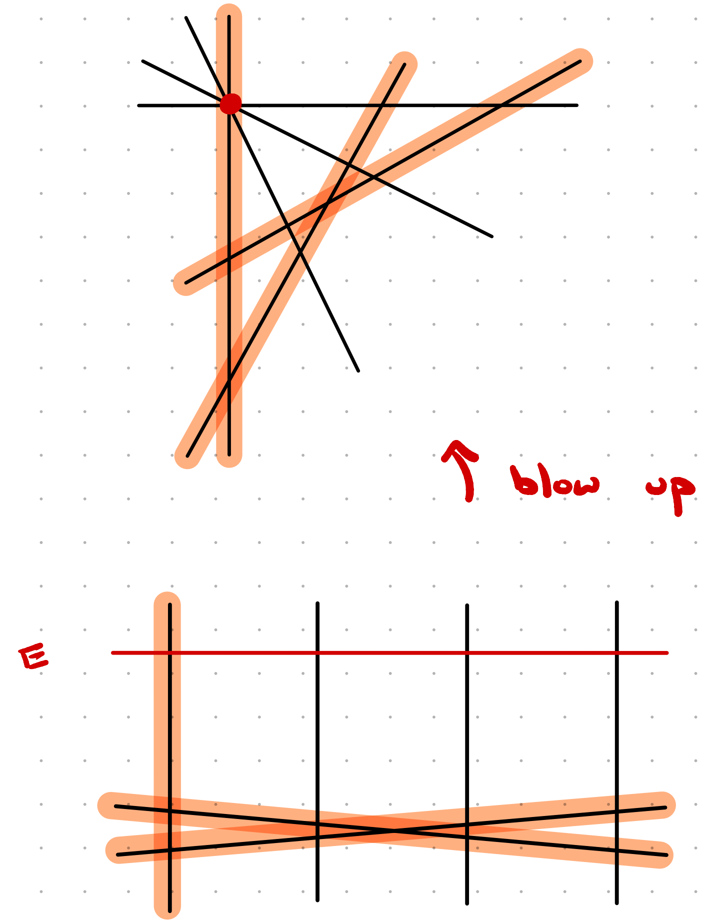
$c(X, B') = 0$

$|\lceil B' \rceil| < |\lceil B \rceil|$.

Example: $(\mathbb{P}^2, B)$, $B =$




blow up
E


blow up
E

New Idea:

Define $\mathcal{A}(X, B) = \{ \Delta \mid (X, \Delta)$ toric log CY, $\lfloor B \rfloor \le \Delta \le \lceil B \rceil \}$.

Induct on $|\mathcal{A}(X, B)|$.

- Given $\Delta \in \mathcal{A}(X, B)$, define λ_1 as before &

$$\lambda_2 = \lambda_2(X, B; \Delta)$$
$$= \sup \{ \lambda \in [0, \lambda_1) \mid (X, B_{\Delta,\lambda} = \frac{B - \lambda\Delta}{1-\lambda}) \text{ is lc} \}$$

4 - Proofs

Lemma 1 If $\lambda_2 < 1$, then

$$\mathcal{A}(x, B_{\Delta, \lambda_2}) \subset \mathcal{A}(x, B)$$

Containment is strict when $\lambda_2 = \lambda_1 < 1$.

Lemma 2 (X,B) logCY of complexity zero.
Let $f: Y \to X$ extract only lc places of (X,B).
Then:

1) log pullback (Y,B_Y) is logCY of compl. zero.

2) $f_* : \mathcal{A}(Y,B_Y) \hookrightarrow \mathcal{A}(X,B)$ is injective
w/ image $\{\Delta \in \mathcal{A}(X,B) \mid E$ lc place for (X,Δ) for all f-exc $E\}$.

Lemma 3 If $\lambda_2 < \lambda_1 < 1$, there is an lc place E of $(X, B_{\Delta,\lambda_2})$ which <u>isn't</u> an lc place of (X,Δ).

Proof of Theorem 1

- Induct on $|\mathcal{A}(X,B)|$
- "Key Argument" Suppose $\Delta \in \mathcal{A}(X,B)$ has $\lambda_1 < 1$.
  - Case 1: $\lambda_2 = \lambda_1 < 1$.

    $\rightsquigarrow$ $(X, B_{\Delta,\lambda_2})$ is log CY of compl zero w/ $|\mathcal{A}(X,B_{\Delta,\lambda_2})| < |\mathcal{A}(X,B)|$.

  - Case 2: $\lambda_2 < \lambda_1 < 1$

    $\rightsquigarrow$ $f: (Y, B_Y) \longrightarrow (X, B_{\Delta,\lambda_2})$ dlt blow-up w/ some f-exc $E \subset Y$ which isn't an lc place for (X,Δ).

    $\rightsquigarrow$ $|\mathcal{A}(Y,B_Y)| < |\mathcal{A}(X,B_{\Delta,\lambda_2})| \leq |\mathcal{A}(X,B)|$

- <u>Base Case ($|\mathcal{A}(X,B)| = 1$)</u>
  - Use "Key Argument" to conclude, by contradiction, $\lambda_1 = 1$.

  $\rightsquigarrow$ $B = \Delta$, (X,B) toric log CY.

- <u>Inductive Step:</u>
  - <u>Case 1:</u> $B_{\Delta,\lambda_2} = \sum_{i=1}^{r} b_i \Delta_i$

    $\rightsquigarrow B = \lambda_2 \Delta + \sum_{i=1}^{r} (1-\lambda_2) b_i \Delta_i$,

  - <u>Case 2:</u> $B_Y = \sum_{i=1}^{r} b_i \Gamma_i$

    $\rightsquigarrow B = \lambda_2 \Delta + \sum_{i=1}^{r} (1-\lambda_2) b_i f_* \Gamma_i$.

$\square$

Corollary

Let (X,B) be a dlt log CY pair of complexity zero. Then $(X, \lfloor B \rfloor)$ is log smooth.

Proof

1) Irred. components of $\mathrm{Sing}(X)$ are toric strata for all toric log CY (X, Δ).

2) Divisors extracted by normalized blow-up of such irred. components are lc places for all toric log CY (X,Δ), hence also for (X,B) by Theorem 1.

$\Rightarrow$ Irred. components of $\mathrm{Sing}(X)$ are lc centers of (X,B).

3) The dlt condition requires all lc centers to intersect the smooth locus.

$\blacksquare$

Main Idea of Theorem 2

- Given (X, B) logCY of complexity zero, X fails to be a generalized Bott tower only when it is singular in "predictable" ways.

- The corollary (& its proof) gives us information about lc centers/places in this case.
  We use this to perform birational modifications that fix our problems while maintaining control over the complexity.